# Holomorphic functions

## Proposition

$H(\Omega)$ is a $\mathbb{C}$-vector space with

1. $f, g \in \mathscr{H}(\Omega) \implies \alpha f + \beta g \in \mathscr{H}(\Omega)$
2. $f, g \in \mathscr{H}(\Omega) \implies fg \in \mathscr{H}(\Omega)$
3. $f, g \in \mathscr{H}(z_0), g(z_0) \neq 0 \implies \frac{f}{g} \in \mathscr{H}(z_0)$
4. $f \in \mathscr{H}(\Omega, U), g \in \mathscr{H}(U) \implies g \circ f \in \mathscr{H}(\Omega)$

## Proposition 2.3

$f(z) = u(x, y) + iv(x, y) \in \mathscr{H}(z_0) \implies$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

or

$$Jf = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}$$

and $|\det Jf| = |f'(z_0)|^2$.

## Theorem 2.4

$f = u + iv, u, v \in C^1 \wedge \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \implies f \in \mathscr{H}(\Omega)$.

## Theorem 2.6

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}(D_R(0)) \qquad f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} \in \mathscr{H}(D_R(0))$$

with $\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n}$.

# Complex line integrals

## Definition: Integral along path

$\gamma : [a, b] \to \mathbb{C}, f \in C^0(\gamma)$:

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

### Proposition

$f, g \in C^0(\gamma_1, \gamma_2 : [a,b] \to \mathbb{C}) \implies$

$$\int_\gamma \alpha f(z) + \beta g(z)\,\mathrm{d}z = \alpha \int_\gamma f(z)\,\mathrm{d}z + \beta \int_\gamma g(z)\,\mathrm{d}z$$
$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z$$
$$\int_{\gamma_1+\gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z$$
$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \sup_{z\in\gamma}|f(z)| L(\gamma)$$

### Theorem 3.2

$f \in C^0(\Omega), \gamma : [a,b] \to \Omega, F' = f \implies$

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

## Cauchy's Theorem and its applications

### Theorem 1.1: Goursat

$f \in \mathscr{H}(\Omega), T \subseteq \Omega, \mathring{T} \subseteq \Omega$ a triangle $\implies$

$$\int_T f(z)\,\mathrm{d}z = 0$$

### Theorem 2.1

$f \in \mathscr{H}(D_r(z_0)) \implies \exists F : F' = f$

### Theorem 2.2b: Cauchy's Theorem for a disc

$f \in C^0(D_r(z_0)), f \in \mathscr{H}(D_r(z_0) \setminus z_1) \implies$

$$\int_\gamma f(z)\,\mathrm{d}z = 0 \qquad \forall \gamma \in D_r(z_0), \gamma(a) = \gamma(b)$$

## Cauchy Integral Formulae

### Theorem 4.1: Cauchy Integral Formula /Theorem 4.4 / Corollary 4.3: Cauchy inequalities

$f \in \mathscr{H}(\Omega \supseteq \overline{D}), C := \partial D$ positive orientation $\implies$

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} \,\mathrm{d}w = f(z) \qquad \forall z \in D$$

$f \in \mathscr{H}(\Omega \supseteq D_r(z_0)) \implies$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} \,\mathrm{d}w \qquad \forall z \in D_r(z_0)$$

$f \in \mathscr{H}(\Omega \supseteq \overline{D_r(z_0)}) \implies$

$$|f^{(n)}(z_0)| \leq \frac{n! \cdot \sup_{|w - z_0| = r} |f(w)|}{r^n}$$

## Theorem/Corollary 4.5: Liouville's Theorem

$f \in \mathscr{H}(\mathbb{C}), \sup_{z \in \mathbb{C}} |f(z)| < \infty \implies f = \text{const.}$

## Definition: Order of a function

$f \in \mathscr{H}(\Omega \ni z_0)$:

$$\operatorname{ord}_{z_0} f := \min\{k \geq 0 \mid f^{(k)}(z_0) \neq 0\}$$

## Proposition 4.6

$f \in \mathscr{H}(\Omega \ni z_0) \implies$

1. $\operatorname{ord}_{z_0} f = \infty \implies f(z) = 0, \forall z \in D_r(z_0)$
2. $\operatorname{ord}_{z_0} f \neq 0 \implies \exists! h \in \mathscr{H}(D_r(z_0)), h(z_0) \neq 0 : f(z) = (z - z_0)^{\operatorname{ord}_{z_0} f} h(z), \forall z \in D_r(z_0)$
3. $\operatorname{ord}_{z_0}(f + g) \geq \min\{\operatorname{ord}_{z_0} f, \operatorname{ord}_{z_0} g\}$ and $\operatorname{ord}_{z_0}(fg) = \operatorname{ord}_{z_0} f + \operatorname{ord}_{z_0} g$

## Theorem 4.8 / Corollary 4.9: Identity theorem / Theorem 4.8b / Corollary 4.9b: Identity theorem / Theorem

$f \in \mathscr{H}(\Omega),$ an infinite set with limit point $z_0 \in \Omega, z_0$ . Then $f(z) = 0, \forall z \in \implies f \;\; 0$
.

$f, g \in \mathscr{H}(\Omega), f(z) = g(z), \forall z \in U \neq \implies f \;\; g.$

$f \in \mathscr{H}(\Omega)$, then the following are equivalent:

1. $f \;\; 0$
2. $\exists z_0 \in \Omega : \operatorname{ord}_{z_0} f = \infty$
3. $\{z \in \Omega \mid f(z) = 0\}$ has a limit point in $\Omega$.

$f, g \in \mathscr{H}(\Omega)$, then the following are equivalent:

1. $f \quad g$
2. $\exists z_0 \in \Omega : f^{(n)}(z_0) = g^{(n)}(z_0), \forall n \geq 0$
3. $\{z \in \Omega \mid f(z) = g(z)\}$ has a limit point in $\Omega$.

$f, g \in \mathscr{H}(\Omega), fg \quad 0 \implies f \quad 0 \quad g \quad 0.$

## Theorem 5.1: Morera's Theorem

Converse to Theorem 1.1 Goursat

$f \in C^0(\Omega), \forall D_r(z_0) \subseteq \Omega, \forall T, \mathring{T} \subseteq D_r(z_0) : {}_T f(z)\,\mathrm{d}z = 0 \implies f \in \mathscr{H}(\Omega).$

# Sequences of holomorphic functions

## Definition: Locally uniformly convergent / Uniformly convergent on compact sets / Proposition / Theorem 5.2 / Theorem 5.3

$f_n : \Omega \to \mathbb{C}$ is called locally uniformly convergent or uniformly convergent on compact sets if the following equivalent hold:

1. $\forall z_0 \in \Omega, \exists \quad 0, D(z_0) \subseteq \Omega : f_n|_{D(z_0)}$ converges uniformly.
2. $\forall \subseteq \Omega$ compact, $f_n|$ converges uniformly.

$f_n \in C^0(\Omega)$ locally uniformly convergent to $f \implies f \in C^0(\Omega)$.

$f_n \in \mathscr{H}(\Omega)$ locally uniformly convergent to $f \implies f \in \mathscr{H}(\Omega)$.

$f_n \in \mathscr{H}(\Omega)$ locally uniformly convergent to $f \implies f_n'$ locally uniformly convergent to $f'$.

## Theorem: Weierstrass -test

$f_n : \Omega \to \mathbb{C}, \neq U \subseteq \Omega$. If $\exists ({}_n)_{n\geq 1} \subseteq , {}_n \geq 0 : |f_n(z)| \leq {}_n, \forall z \in U, \forall n, {}_n < \infty \implies$

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly on $U$.

## Proposition 2.1: Riemann Zeta / Example

$$() := \sum_{n=1}^{\infty} \frac{1}{n}$$

converges absolutely and uniformly on $U := \{ \in \mathbb{C} \mid \mathrm{e} \ \geq 1 + \}, \forall \ 0$ and $\in \mathscr{H}(\{ \in \mathbb{C} \mid \mathrm{e} \ 1\})$.

$z \in \ := \{z \in \mathbb{C} \mid \mathrm{m}\, z \ 0\}$:

$$(z) := \sum_{n \in} {}^{2\pi i n^2 z} = 1 + 2 \sum_{n=1}^{\infty} {}^{2\pi i n^2 z}$$

## Definition: Isolated singularity / Singularity types

$z_0 \in \mathbb{C}$ is a possible isolated singularity of $f$ if $\exists r \ 0 : f \in \mathscr{H}(D_r(z_0))$.

Singularities:

1. Removable: can be extended holomorphically: $\frac{\sin z}{z}$
2. Pole: Real singularity: $\frac{1}{z}$
3. Essential singularity: ${}^{1/z}$

## Definition: Removable singularity / Theorem: Riemann's continuation theorem

$f \in \mathscr{H}(\Omega \setminus \{z_0\}), z_0$ is a removable singularity if $\exists f \in \mathscr{H}(\Omega) : f(z) = f(z), \forall z \in \Omega \setminus \{z_0\}$.

$f \in \mathscr{H}(\Omega \setminus \{z_0\}), z_0$ is a removable singularity if the following equivalent hold:

1. $f$ is holomorphically extendable to $\Omega$
2. $f$ is continuously extendable to $\Omega$
3. $\exists r \ 0$ : $f$ is bounded in $D_r(z_0)$
4. $\lim_{z \to z_0} (z - z_0) f(z) = 0$

## Definition: Pole

If $\exists n \in \ : (z - z_0)^n f(z)$ is bounded near $z_0$, then $z_0$ is a pole of $f$ with the order of the pole $\ := \min\{n \ \}$.

## Theorem 1.2b

$\in \ , f \in \mathscr{H}(\Omega \setminus \{z_0\})$, the following are equivalent:

1. $f$ has a pole of order  at $z_0$

2. $\exists r \;\; 0, g \in \mathscr{H}(D_r(z_0)), g(z_0) \neq 0 : f(z) = \frac{g(z)}{(z-z_0)}, \forall z \in D_r(z_0)$
3. $\exists r \;\; 0, h \in \mathscr{H}(D_r(z_0)), h(z) \neq 0, \forall z \in D_r(z_0) : f(z) = \frac{1}{h(z)}$ where $\operatorname{ord}_{z_0} h =$

# Theorem 1.3 / Theorem 1.4 / Lemma

$f$ has a pole of order $n$ at $z_0$, then

$$f(z) = \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \; + \overbrace{a_{-1}}^{=:\operatorname{es}_{z_0} f}\frac{}{z - z_0}}_{=:_{z_0}(f,z)} + \underbrace{(z)}_{\in \mathscr{H}(D_r(z_0))}$$

$$\operatorname{es}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}((z-z_0)^n f(z))$$

$f, g \in \mathscr{H}(z_0), \operatorname{ord}_{z_0} g = 1 \implies \frac{f}{g}$ has a simple pole with

$$\operatorname{es}_{z_0} \frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$$

# Theorem 2.1: Residue formula

$F = \{z_0, \;, z_n\}, f \in \mathscr{H}(\Omega \setminus F)$ with poles in $F, \gamma = \partial D$ positive in $\Omega, \gamma \;\; F = \;\; \implies$

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{z_i \in FD} \operatorname{es}_{z_i} f$$

# Integral solution methods

## 1

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$$

choose

$$f(z) = \frac{1}{1+z^2}$$

and a contour with the top half circle of radius $R$ and let $R \to \infty$, bounding other parts of the integral.

## 2

$$\int_{-\infty}^{\infty} \frac{(x)}{(x)}\,\mathrm{d}x$$

where  has no zeros on the real line. For $\partial \geq \partial + 2$ we get

$$\int_{R} \frac{(z)}{(z)} \,\mathrm{d}z \overset{R\to\infty}{} \quad 0$$

for $_R$ the top half circle.

**4**

$$\int_{-\infty}^{\infty} \frac{(x)}{(x)} \cos(ax) \,\mathrm{d}x$$

choose

$$f(z) = \frac{(z)}{(z)} {}^{iaz}$$

such that $|^{iz}| \leq 1$ where $\mathrm{m}\, z \;\; 0$.

**5**

$$\int_{0}^{2\pi} \frac{(\cos t, \sin t)}{(\cos t, \sin t)} \,\mathrm{d}t$$

where  has no zeros on $x^2 + y^2 = 1$. Write $\cos \;\, = \frac{1}{2}\left(z + \frac{1}{z}\right), \sin z = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ with $\frac{\mathrm{d}z}{iz} = \mathrm{d}$ to solve.

## Proposition/Corollary 3.2

$z_0$ is a pole of $f$

$$\lim_{z\to z_0} |f(z)| = \infty$$

## Theorem: Casorati-Weierstrass / Picard 1879

$f \in \mathscr{H}(D_r(z_0)), z_0$ is an essential singularity, then $f(D_r(z_0))$ is dense in $\mathbb{C}$.

$f \in \mathscr{H}(D_r(z_0)), z_0$ is an essential singularity, then $|\mathbb{C} \setminus f(D_r(z_0))| \leq 1$. (Example: for $f(z) = {}^{1/z} \implies \mathbb{C} \setminus f(D_r(0)) = \{0\}$.)

# Meromorphic functions

## Definition: Extended complex plane

$\mathbb{C} := \mathbb{C} \;\; \{\infty\}$ with

- $z \;\; \infty = \infty, \forall z \in \mathbb{C}$

- $z \cdot \infty = \infty, \forall z \in \mathbb{C} \setminus \{0\}$
- $\frac{z}{\infty} = 0, \forall z \in \mathbb{C}$
- $\frac{z}{0} = \infty, \forall z \in \mathbb{C} \setminus \{0\}$

## Definition: Meromorphic function / Proposition / Proposition

$f : \Omega \to \mathbb{C}$ is $f \in (\Omega)$ if

1. $_f := \{z \in \Omega \mid f(z) = \infty\} = f^{-1}(\{\infty\})$ has no limit point in $\Omega$.
2. $_f$ contains the poles of $f$.
3. $f|_{\Omega \setminus_f} \in \mathscr{H}(\Omega)$
4. $\mathscr{H}(\Omega) \subseteq (\Omega)$
5. $f, g \in (\Omega) \implies \alpha f + \beta g \in (\Omega)$, or $(\Omega)$ is a vector space.
6. $f, g \in (\Omega), z_0 \in {}_f \ {}_g, f = {}_f + f, g = {}_g + g, f, g \in \mathscr{H}(\Omega)$
   $\implies fg = ({}_f + f)({}_g + g) = {}_{fg} + , \in \mathscr{H}(\Omega)$
7. $0 \ \ f \in (\Omega)$ and the zeros do not have a limit point in $\Omega$, then $\frac{1}{f} \in (\Omega)$

$0 \ \ f \in (\Omega)$ open, connected, then

$$:= \{z \in \Omega \mid f(z) = 0\}$$

has no limit point in $\Omega$.

## Definition: Order/Valuation of a function / Proposition

Generalization of Definition Order of a function

$0 \ \ f \in (\Omega \ni z_0), \operatorname{ord}_{z_0} f = k$:

1. $f(z_0) \neq \infty \implies k \geq 0$ is the order of the zero at $z_0$.
2. $f(z_0) = \infty \implies k \leq -1$ is the negative order of the pole at $z_0$.

$0 \ \ f \in (\Omega \ni z_0) \implies$

1. $k = \operatorname{ord}_{z_0} f \quad \exists r \ \ 0, h \in \mathscr{H}(D_r(z_0)) : h(z_0) \neq 0, f(z) = (z - z_0)^k h(z)$
2. $\operatorname{ord}_{z_0}(fg) = \operatorname{ord}_{z_0} f + \operatorname{ord}_{z_0} g$
3. $f + g \neq 0 \implies \operatorname{ord}_{z_0}(f + g) \geq \min\{\operatorname{ord}_{z_0} f, \operatorname{ord}_{z_0} g\}$

## Definition: Singularities at $\infty$

$f$ analytic for $|z| \ \ \frac{1}{R}, R \ \ 0$ has an isolated singularity at $\infty$ if

$$g(z) := f\left(\frac{1}{z}\right)$$

has an isolated singularity at $z = 0$. $f \in (\mathbb{C})$ and $\mathscr{H}(\infty)$ or has a pole at $\infty \implies (\mathbb{C})$.

### Theorem

$f \in (\mathbb{C}) \implies$

$$f(z) = \frac{(z)}{(z)} \qquad (z), (z) \in \mathbb{C}[z]$$

## Application of the Residue theorem

### Lemma

$0 \;\; f \in (\Omega \ni z_0), \Omega$ open, connected $\implies$ logarithmic derivative of $f$ : $\frac{f'}{f} \in (\Omega)$ and has simple poles at $z_0$ if $\operatorname{ord}_{z_0} f \neq 0$ with

$$\operatorname{es}_{z_0} \frac{f'}{f} = \operatorname{ord}_{z_0} f$$

### Theorem 4.1: Argument principle

$\Omega$ open, connected, $\gamma$ closed with interior  such that residue theorem applies. $f$ has no zeros or poles on $\gamma \implies$

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z = \sum_{\substack{z_0 \in \\ \operatorname{ord}_{z_0} f \neq 0}} \operatorname{ord}_{z_0} f = \; -$$

where  is the number of zeros,  the number of poles with multiplicity inside .

### Theorem 4.3: Rouché

$f, g \in \mathscr{H}(\Omega \supseteq \overline{D_r(z_0)}), |f(z)| \;\; |g(z)|, \forall z \in C_r(z_0) \implies f, f+g$ have the same number of zeros inside $C$.

### Theorem 4.4: Open mapping theorem

$\text{const.} \neq f \in \mathscr{H}(\Omega)$ open, connected, then $f$ is open ($f(U)$ open is open).

### Theorem 4.5 / Corollary 4.6: Maximum modulus principle

$\text{const.} \neq f \in \mathscr{H}(\Omega)$ open, connected $\implies$

$$z_0 \in \Omega : |f(z)| \leq |f(z_0)| \qquad \forall z \in \Omega$$

or $f$ cannot attain a maximum in $\Omega$. In particular: $\overline{\Omega}$ bounded, $f \in C^0(\Omega) \implies$

$$\underset{z\in\overline{\Omega}}{\mathrm{m}}|f(z)| = \underset{z\in\partial\Omega}{\mathrm{m}}|f(z)|$$

exists, since $f$ is continuous on the bounded closed set $\overline{\Omega}$.

# Homotopy and simply connected domains

## Definition: Homotopy

$\gamma_0, \gamma_1 : [a,b] \to \Omega, \gamma_0(a) = \gamma_1(a), \gamma_0(b) = \gamma_1(b)$ are homotopic in $\Omega$ if

$$\begin{aligned} \exists H : [a,b]\ \ [0,1] &\to \Omega \\ (t, )\ \ &H(t, ) \end{aligned}$$

continuous with

1. $H(t, ) = \gamma(t)$
2. $\gamma(t) := H(t, ) \in C^0([a,b])$ with the same endpoints

## Theorem 5.1: Homotopy theorem

$\gamma_0, \gamma_1 : \ \to \Omega, \gamma_0 \ {}_{\Omega}\ \gamma_1, f \in \mathscr{H}(\Omega) \implies$

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

## Definition: Simply connected

$\Omega$ is simply connected if every two curves with the same endpoints are homotopic.

## Theorem 5.2

$f \in \mathscr{H}(\Omega)$ simply connected has a primitive with $\gamma$ closed $\implies$

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0$$

and any two primitives differ by a constant.

# The complex logarithm

## Definition: Branch of logarithm / Remark

$\mathrm{lo}_{\Omega} \in \mathscr{H}(\Omega)$:

$$\mathrm{ep}(\mathrm{lo}_\Omega z) = z$$

1. ${}^{z} \neq 0 \implies 0 \ \ \Omega$ is necessary
2. $\Omega = \mathbb{C} \setminus \{0\}$ does not work:
   $\mathrm{ep}(f(z)) = z \implies f'(z)\,\mathrm{ep}(f(z)) = 1 \implies f'(z) = \frac{1}{z}$ does not give $0$ integrated over every closed path
3. Two logarithms ,   differ by
   $\ - \ \ = 2\pi i n, n \in \ \ : \mathrm{ep}((z) - \ (z)) = 1 \implies (z) - \ (z) \in 2\pi i$ constant

## Theorem 6.1

$\Omega \subseteq \mathbb{C} \setminus \{0\}$ simply connected $\implies \exists F \in \mathscr{H}(\Omega) : \mathrm{ep}(F(z)) = z$ branch of logarithm.

## Definition: Principal branch of logarithm / Proposition / Remark

$\Omega = \mathbb{C}^- := \mathbb{C} \setminus (-\infty, 0]$:

$$\mathrm{o} := \mathrm{lo}_\Omega \qquad \wedge \qquad \mathrm{lo}_\Omega(1) = 0$$

$z = r^{i} \in \mathbb{C}^-, r \ \ 0, -\pi < \ \ < \pi \implies$

$$\mathrm{o}\, z = \mathrm{lo}\, r + i$$

$\mathrm{lo}\, z + \mathrm{lo}\, w = \mathrm{lo}\, zw$ does not hold in general: $w = r^{i\alpha}, z = {}^{i\beta}, zw = r^{i}$ with $\alpha, \beta, \ \in (-\pi, \pi) \implies \ \ = \alpha + \beta + \gamma, \gamma \in \{-2\pi, 0, 2\pi\}$ thus
$\mathrm{lo}\, zw = \mathrm{lo}\, z + \mathrm{lo}\, w \quad \gamma = 0 \quad \alpha + \beta \in (-\pi, \pi)$

$\mathrm{o}(\{|z| = r \mid -\pi < \mathrm{r}\, z < \pi\}) = \{\mathrm{e}\, w = \mathrm{lo}|z|, -\pi < \mathrm{m}\, w < \pi\}$

$\mathrm{o}(\{z \mid \mathrm{r}\, z = \ \}) = \{w \mid \mathrm{m}\, w = \ \}$

We can define a branch of logarithm for any $\Omega = \mathbb{C} \setminus (\{z \mid \mathrm{r}\, z = \alpha\} \ \ \{0\})$.

## Definition: Complex power

$\mathrm{lo}_\Omega$ branch of logarithm:

$$z^\alpha := \mathrm{ep}(\alpha\, \mathrm{lo}_\Omega z)$$

This depends on $\mathrm{lo}_\Omega : \mathrm{lo}_\Omega \ \ \mathrm{lo}_\Omega + 2\pi i k \implies$

$$\mathrm{ep}(\alpha(\mathrm{lo}_\Omega z + 2\pi i k)) = z^{\alpha\, 2\pi i k \alpha}$$

Principal branch of logarithm:

$$(z^{1/}) = \mathrm{ep}\left(\frac{1}{}\mathrm{lo}\, z\right)\mathrm{ep}\left(\frac{1}{}\mathrm{lo}\, z\right) = \mathrm{ep}\left(\text{-}\mathrm{lo}\, z\right) = z$$

## Theorem 6.2 / Corollary

$f \in \mathscr{H}(\Omega)$ simply connected, $f(z) \neq 0, \forall z \in \Omega \implies$

$$\exists g \in \mathscr{H}(\Omega) : \mathrm{ep}(g(z)) = f(z)$$

called the logarithm of $f$ and

$$\exists h \in \mathscr{H}(\Omega) : h^2(z) = f(z)$$

called the square root of $f$.

## Proposition

$f \in (\Omega),\ := \Omega \setminus {}_f : f \in \mathscr{H}(), \gamma_1\ {}_\Omega\ \gamma_2 \implies$

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z$$

and if $\gamma_2$ satisfies residue theorem with interior $\implies$

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = 2\pi i \sum_{w \in} \mathrm{es}_w\, f$$

## Definition: Winding number

$\mathbb{C} \ni z_0\ \ \gamma$ closed:

$${}_\gamma(z_0) = \mathrm{ind}_\gamma\, z_0 := \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,\mathrm{d}z$$

## Proposition 1.3

$\gamma$ closed, ${}_\gamma : \Omega = \mathbb{C} \setminus \mathrm{im}\,\gamma \to \mathbb{C}$ continuous takes values in $\implies$ is constant on any connected subset of $\Omega$ with ${}_\gamma(z) = 0, |z|$ large enough.

## Theorem: Residue formula

Generalization of Theorem 2.1 Residue formula

$f \in (\Omega)$ simply connected, $\gamma$ closed in $= \Omega \setminus {}_f \implies$

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{z_0 \in {}_f} {}_\gamma(z_0)\,\mathrm{es}_{z_0}\, f$$

# Conformal maps

## Definition: Conformal map

$U$, open, $f : U \to$ injective, holomorphic is a conformal map.

$f$ bijective conformal map is a conformal equivalence / biholomorphic / holomorphic isomorphism.

## Proposition 1.1 / Remark / Corollary

$f : U \to$ conformal $\implies$

$$f'(z) \neq 0 \qquad \forall z \in U$$

and $f^{-1} : \operatorname{im} f \subseteq \; \to U$ is holomorphic.

Thus $f : U \to$ conformal equivalence $\quad f^{-1}$ conformal equivalence (conformal equivalence is an equivalence relation).

$f : U \to$ conformal equivalence, then

$$T : \mathscr{H}() \to \mathscr{H}(U)$$
$$\circ f$$

is a linear isomorphism of vector spaces.

## Examples

- $f : \; \to , z \;\; \frac{z-i}{z+i}$ with $f^{-1} : w \;\; i\frac{1+w}{1-w}$
- $f : \left\{z \in \mathbb{C} \mid 0 < \mathrm{r}\, z < \frac{\pi}{n}\right\} \to , z \;\; z^n$ with $f^{-1} : w \;\; w^{1/n}$
- $\mathrm{o} : \mathbb{C}^- \to \{z \in \mathbb{C} \mid \mathrm{e}\, z \;\; 0 \wedge -\pi < \mathrm{m}\, z < \pi\}$ with $\mathrm{o}^{-1} = \mathrm{ep}$
- $\mathbb{C} \;_C$ , since a map $\mathbb{C} \to$ entire would be bounded and thus constant (Theorem/Corollary 4.5 Liouville's Theorem)

# Riemann mapping theorem

## Theorem 8.3.1: Riemann mapping theorem / Corollary

$\neq \Omega \;\; \mathbb{C}$ simply connected, $z_0 \in \mathbb{C} \implies \exists!$ conformal equivalence

$$F : \Omega \to$$
$$F(z_0) = 0$$
$$F'(z_0) \in (0, \infty) \subseteq \mathbb{C}$$

Any simply connected proper subsets of $\mathbb{C}$ are conformally equivalent.

## Theorem 2.2 (Step 1) / Corollary

$f: \to$ automorphism $\implies \exists \in , \alpha \in$ :

$$f(z) = {}^{i} \frac{\alpha - z}{1 - \overline{\alpha} z}$$
$$f(0) = {}^{i}\alpha$$
$$f'(0) = {}^{i}(|\alpha|^2 - 1)$$

and every map of this form is an automorphism of .

The map in Theorem 8.3.1 Riemann mapping theorem / Corollary is unique:
$f_i : \Omega \to , f_i(z_0) = 0, f_i'(z_0) \;\; 0 \implies f_1 = f_2$.

## Lemma: Schwarz

$f \in \mathscr{H}(,), f(0) = 0 \implies$

1. $|f(z)| \leq |z|, \forall z \in$
2. $\exists z_0 \neq 0 : |f(z_0)| = |z_0| \implies \exists \in \; : f(z) = {}^{i} z$
3. $|f'(0)| \leq 1$ with equality $\quad f(z) = {}^{i} z$

## Theorem 2.4

$g: \to$ automorphism $\implies$

$$g(z) = \frac{az + b}{z +} \qquad := \begin{pmatrix} a & b \\ & \end{pmatrix} \in L_2()$$

with $\det \;\; 0$.

## Proposition (Step 2)

$\neq \Omega \;\; \mathbb{C}$ open, connected $\implies \exists f : \Omega \to$ conformal with $0 \in f(\Omega)$ (or: $\Omega$ is conformally equivalent to a subset of ).

## Lemma

$:= \{f : \Omega \to \; | \; \mathrm{conorml} \wedge f(z_0) = 0\} \implies$

$$:= \sup_{f \in} |f'(z_0)| < \infty$$

## Key Proposition

$\exists f \in \; : |f'(z_0)| =$

## Proposition (Step 4)

$f \in$ with

$$|f'(z_0)| = \ = \sup_{f\in} |f'(z_0)|$$

is a conformal equivalence.

## Theorem: Montel

$(f_n)_n \subseteq \mathscr{H}(\Omega), \forall \subseteq \Omega$ compact, $\exists_k \ 0 : |f_n(z)| < {}_k, \forall z \in \implies \exists (f_{n_k})$ converging uniformly on compact sets.

## Proposition

$(f_n)_n \subseteq , f_n \to f, \forall z \in \Omega$ uniformly on compact sets
$\implies f = \text{const}. \ f \in \ : \lim_{n\to\infty} f_n'(z_0) = f'(z_0).$

## Lemma

$\Omega$ open, connected, $f_n : \Omega \to$ conformal, $f_n \to f$ uniformly on compact sets
$\implies f = \text{const}.$ or injective.